

On Microscopic Origin of the Fokker – Planck Kinetic Evolution of Hard Spheres

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Abstract. The rigorous approach to the description of the kinetic evolution of a many-particle system composed of a trace hard sphere and an environment of finitely many hard spheres is developed. We prove that the evolution of states of a trace hard sphere in an environment can be described within the framework of the marginal distribution function governed by the generalized Fokker – Planck kinetic equation and an infinite sequence of the explicitly defined functionals of this function.

Key words: Fokker-Planck equation; kinetic equation; cluster expansion; scattering operator; cumulant of groups of operators; scaling limit; colliding particles.

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1 Introduction

The rigorous derivation of kinetic equations for many-particle systems composed of a trace particle moving in an environment of particles, particularly the Fokker – Planck kinetic equation, remains an open problem so far. It should be noted there are wide applications of the Fokker – Planck equation to the description of kinetic processes of various nature [1–4].

As is known, the Fokker – Planck kinetic equation was stated in papers [5], [6] by the instrumentality of phenomenological treatment. The consistent microscopic derivation of the Fokker – Planck equation on the basis of methods of the perturbation theory springs from works of N.N. Bogolyubov [7], [8]. In these works the nature of a stochasticity into deterministic systems was elucidated for the first time.

In modern research a main approach to the problem of the rigorous derivation of the Fokker – Planck kinetic equation lies in the construction of the scaling (diffusion) limit [9] of a solution of evolution equations which describe the evolution of states of a many-particle system composed of a trace particle and an environment, in particular, a perturbative solution of the corresponding BBGKY hierarchy [10]. The rigorous results on the justification of the Fokker – Planck kinetic equation in scaling limits for particles interacting as hard spheres was obtained in papers [11], [12]. The review of recent results, including quantum systems, was given in article [13].

In this paper we develop a rigorous approach to the description of the kinetic evolution of a many-particle system composed of a trace hard sphere and an environment of finitely many hard spheres. On the basis of the stated kinetic cluster expansions of the cumulants of groups of operators, which are the generating evolution operators of a nonperturbative solution of the BBGKY hierarchy, we prove that all possible states of a trace hard sphere in an environment at

arbitrary moment of time can be described within the framework of the marginal distribution function of the trace hard sphere governed by the generalized Fokker – Planck kinetic equation and the explicitly defined functionals of this function without any approximations. Thus, we establish that the stated Fokker – Planck kinetic equation gives an alternative approach for the description of the evolution of states of a trace particle in an environment. We remark that the specific Fokker-Planck-type kinetic equations can be derived from the constructed generalized Fokker-Planck kinetic equation in the appropriate scaling limits or as a result of certain approximations.

We briefly outline the structure of the paper. In sections 2 we formulate necessary preliminary facts about dynamics of a trace hard sphere in an environment. In sections 3 the main results related to the origin of the Fokker – Planck kinetic evolution are stated. Then in sections 4-6 the main results are proved, in particular in section 5, using the kinetic cluster expansions of cumulants of operators stated in section 4, we derive the generalized Fokker – Planck kinetic equation. Finally, in section 7 we conclude with some observations and perspectives for future research.

2 The evolution of a trace hard sphere in an environment

We consider a many-particle system composed of a trace particle and an environment which is a system of a non-fixed, i.e. arbitrary, but finite number of identical particles in the space \mathbb{R}^3 . If the environment is in the equilibrium state, for such a system it is used the term a trace particle in a heat bath (or in a thermostat) [11].

We assume that particles are elastically interacting hard spheres with a diameter $\sigma > 0$. Let the trace hard sphere with the mass M be characterized by the phase coordinates $(q, p) \equiv x \in \mathbb{R}^3 \times \mathbb{R}^3$, and identical hard spheres with the mass m of the environment be characterized by the phase coordinates $(q_i, p_i) \equiv x_i \in \mathbb{R}^3 \times \mathbb{R}^3$, $i \geq 1$. For configurations of such a system the set $\mathbb{W}_{1+n} \equiv \{(q, q_1, \dots, q_n) \in \mathbb{R}^{3(1+n)} \mid |q_i - q_j| < \sigma \text{ for at least one pair } (i, j) : i \neq j \in (1, \dots, n) \text{ and } |q - q_j| < \sigma, \text{ if } j \in (1, \dots, n)\}$ is the set of forbidden configurations.

The evolution of all possible states of the trace hard sphere in the environment is described by the sequence of marginal distribution functions $F(t) = (1, F_{1+0}(t, x), F_{1+1}(t, x, x_1), \dots, F_{1+s}(t, x, x_1, \dots, x_s), \dots)$, that satisfy the initial-value problem of the BBGKY hierarchy

$$\frac{\partial}{\partial t} F_{1+0}(t) = \mathcal{L}_{1+0} F_{1+0}(t) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \mathcal{L}_{\text{int}}(\mathbf{t}, 1) F_{1+1}(t), \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial t} F_{1+s}(t) &= \mathcal{L}_{1+s} F_{1+s}(t) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \mathcal{L}_{\text{int}}(\mathbf{t}, s+1) F_{1+s+1}(t) + \\ &+ \sum_{i=1}^s \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \mathcal{L}_{\text{int}}(i, s+1) F_{1+s+1}(t), \end{aligned}$$

$$F_{1+s}(t)|_{t=0} = F_{1+s}^0, \quad s \geq 0. \quad (2)$$

If $t \geq 0$, in hierarchy of evolution equations (1) the operator \mathcal{L}_{1+s} is defined by the Poisson

bracket of noninteracting particles with the corresponding boundary conditions on $\partial\mathbb{W}_{1+s}$ [10]:

$$\begin{aligned} \mathcal{L}_{1+s}F_{1+s}(t) \doteq & -\left\langle \frac{p}{M}, \frac{\partial}{\partial q} \right\rangle_{|\partial\mathbb{W}_{1+s}} F_{1+s}(t, x, x_1, \dots, x_s) - \\ & - \sum_{i=1}^s \left\langle \frac{p_i}{m}, \frac{\partial}{\partial q_i} \right\rangle_{|\partial\mathbb{W}_{1+s}} F_{1+s}(t, x, x_1, \dots, x_s), \quad s \geq 0, \end{aligned} \quad (3)$$

where we denote a scalar product by $\langle \cdot, \cdot \rangle$. The operators $\mathcal{L}_{\text{int}}(i, s+1)$ and $\mathcal{L}_{\text{int}}(\mathbf{t}, s+1)$ are defined by the following expressions:

$$\begin{aligned} \sum_{i=1}^s \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \mathcal{L}_{\text{int}}(i, s+1) F_{1+s+1}(t) \doteq & \sigma^2 \sum_{i=1}^s \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} dp_{s+1} d\eta \langle \eta, \left(\frac{p_i}{m} - \frac{p_{s+1}}{m} \right) \rangle \times \\ & (F_{1+s+1}(t, x, x_1, \dots, q_i, p_i^*, \dots, x_s, q_i - \sigma\eta, p_{s+1}^*) - F_{1+s+1}(t, x, x_1, \dots, x_s, q_i + \sigma\eta, p_{s+1})), \\ \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \mathcal{L}_{\text{int}}(\mathbf{t}, s+1) F_{1+s+1}(t) \doteq & \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_{s+1} d\eta \langle \eta, \left(\frac{p}{M} - \frac{p_{s+1}}{m} \right) \rangle \times \\ & (F_{1+s+1}(t, q, p^*, x_1, \dots, x_s, q - \sigma\eta, p_{s+1}^*) - F_{1+s+1}(t, x, x_1, \dots, x_s, q + \sigma\eta, p_{s+1})), \end{aligned} \quad (4)$$

where $\mathbb{S}_+^2 \doteq \{\eta \in \mathbb{R}^3 \mid |\eta| = 1, \langle \eta, (p_i - p_{s+1}) \rangle > 0\}$ and $\mathbb{S}_{0,+}^2 \doteq \{\eta \in \mathbb{R}^3 \mid |\eta| = 1, \langle \eta, \left(\frac{p}{M} - \frac{p_{s+1}}{m} \right) \rangle > 0\}$. The momenta p_i^* , p_{s+1}^* and p^* , p_{s+1}^* are defined by equalities:

$$\begin{aligned} p_i^* & \doteq p_i - \eta \langle \eta, (p_i - p_{s+1}) \rangle, \\ p_{s+1}^* & \doteq p_{s+1} + \eta \langle \eta, (p_i - p_{s+1}) \rangle; \\ p^* & \doteq p - \frac{2Mm}{M+m} \eta \langle \eta, \left(\frac{p}{M} - \frac{p_{s+1}}{m} \right) \rangle, \\ p_{s+1}^* & \doteq p_{s+1} + \frac{2Mm}{M+m} \eta \langle \eta, \left(\frac{p}{M} - \frac{p_{s+1}}{m} \right) \rangle. \end{aligned} \quad (5)$$

If $t \leq 0$, the BBGKY hierarchy generator is defined by the corresponding operator [10].

Further we consider initial data (2) of statistically independent a trace hard sphere and hard spheres of an environment, i.e. at initial instant the marginal distribution functions satisfy the condition (a chaos property [10])

$$F_{1+s}(t)|_{t=0} = F_{1+0}^0(x) F_{0+s}^0(x_1, \dots, x_s) \prod_{i=1}^s \mathcal{X}_2(q, q_i), \quad s \geq 0, \quad (6)$$

where $\mathcal{X}_2(q, q_i)$ is the Heaviside step function of allowed configurations $\mathbb{R}^6 \setminus \mathbb{W}_2$.

To construct a solution of initial-value problem (1)-(6) we shall adduce some preliminaries about dynamics of the examined system. Let $L_{1+n}^1 \equiv L^1(\mathbb{R}^{3(1+n)} \times (\mathbb{R}^{3(1+n)} \setminus \mathbb{W}_{1+n}))$ be the space of integrable functions f_{1+n} defined on the phase space of $1+n$ particles that are symmetric with respect to the permutations of the arguments x_1, \dots, x_n , nonsymmetric with respect to the permutations of the argument x and the arguments x_1, \dots, x_n , and equal to zero on the set

of forbidden configurations \mathbb{W}_{1+n} . We denote by $L_{1+n,0}^1 \subset L_{1+n}^1$ the subspace of continuously differentiable functions with compact supports.

On a set of measurable functions f_{1+n} defined on the phase space $\mathbb{R}^{3(1+n)} \times (\mathbb{R}^{3(1+n)} \setminus \mathbb{W}_{1+n})$ the following one-parameter mapping: $\mathbb{R} \ni t \mapsto S_{1+n}(-t)f_{1+n}$, is defined by the formula:

$$\begin{aligned} S_{1+n}(-t, \mathbf{t}, 1, \dots, n)f_{1+n}(x, x_1, \dots, x_n) &\doteq \\ &\doteq \begin{cases} f_{1+n}(\mathbf{X}(-t, x, x_1, \dots, x_n), \mathbf{X}_1(-t, x, x_1, \dots, x_n), \dots, \mathbf{X}_n(-t, x, x_1, \dots, x_n)), \\ (x, x_1, \dots, x_n) \in (\mathbb{R}^{3(1+n)} \times (\mathbb{R}^{3(1+n)} \setminus \mathbb{W}_{1+n})) \setminus \mathcal{M}_{1+n}^0, \\ 0, (q, q_1, \dots, q_n) \in \mathbb{W}_{1+n}, \end{cases} \end{aligned} \quad (7)$$

where $\mathbf{X}_i(-t)$ is a phase trajectory of i th hard sphere of an environment and $\mathbf{X}(-t)$ is a phase trajectory of a trace hard sphere constructed in [10]. We note that phase trajectories of a hard sphere system are defined almost everywhere on the phase space $\mathbb{R}^{3(1+n)} \times (\mathbb{R}^{3(1+n)} \setminus \mathbb{W}_{1+n})$ beyond the set \mathbb{M}_{1+n}^0 of the zero Lebesgue measure [10].

A generator of the isometric group of operators (7) is the Liouville operator defined by (3).

If $F_{1+0}^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $F_{0+s}^0 \in L^1(\mathbb{R}^{3s} \times (\mathbb{R}^{3s} \setminus \mathbb{W}_s))$, then a nonperturbative solution of initial-value problem (1),(6) is a sequence of the distribution functions $F_{1+s}(t, x, x_1, \dots, x_s)$, $s \geq 0$, represented by the following series:

$$\begin{aligned} F_{1+s}(t, x, x_1, \dots, x_s) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \dots dx_{s+n} \mathfrak{A}_{1+n}(-t, \{\mathbf{t}, Y\}, X \setminus Y) F_{1+0}^0(x) \times \\ &\times F_{0+s+n}^0(x_1, \dots, x_{s+n}) \prod_{i=1}^{s+n} \mathcal{X}_2(q, q_i), \end{aligned} \quad (8)$$

where the generating evolution operator $\mathfrak{A}_{1+n}(-t)$ is the $(n+1)$ th-order cumulant of groups of operators (7):

$$\mathfrak{A}_{1+n}(-t, \{\mathbf{t}, Y\}, X \setminus Y) = \sum_{\mathbf{P}: \{\{\mathbf{t}, Y\}, X \setminus Y\} = \bigcup_i X_i} (-1)^{|\mathbf{P}|-1} (|\mathbf{P}| - 1)! \prod_{X_i \subset \mathbf{P}} S_{|\theta(X_i)|}(-t, \theta(X_i)), \quad (9)$$

and the following notations are used: $\{\mathbf{t}, Y\}$ is a set consisting of one element (\mathbf{t}, Y) , $Y \equiv (1, \dots, s)$, i.e. $|\{\mathbf{t}, Y\}| = 1$, $\sum_{\mathbf{P}}$ is a sum over all possible partitions \mathbf{P} of the set $(\{\mathbf{t}, Y\}, X \setminus Y) \equiv (\{\mathbf{t}, Y\}, s+1, \dots, s+n)$ into $|\mathbf{P}|$ nonempty mutually disjoint subsets $X_i \in (\{\mathbf{t}, Y\}, X \setminus Y)$, the mapping θ is the declusterization mapping defined by the formula: $\theta(\{Y\}, X \setminus Y) = X$.

Let $F_{1+0}^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and initial distribution functions of an environment belong to the space of integrable functions such that: $\sup_{n \geq 0} \alpha^{-n} \|F_{0+n}^0\|_{L_n^1} < +\infty$, where $\alpha > 0$ is a parameter. Then under the condition that: $\alpha < e^{-1}$, series (8) converges in the norm of the space L_{1+s} for arbitrary $t \in \mathbb{R}^1$. If $F_{1+0}^0 \in L_{1,0}^1$ and $F_{0+s+n}^0 \in L_{s+n,0}^1$, the sequence of functions (8) is a strong solution of initial-value problem of the BBGKY hierarchy (1),(6) and for arbitrary initial data it is a weak solution [10].

In consequence of the fact that initial data of a many-particle system composed of a trace particle and an environment is specified by the initial marginal distribution function of a trace particle, the initial-value problem of the BBGKY hierarchy (1),(6) is not completely well-defined Cauchy problem, because the generic initial data, is not independent for every unknown

marginal distribution function from the hierarchy of evolution equations. Consequently such initial-value problem can be naturally reformulated as the new Cauchy problem of the kinetic equation for the marginal distribution function of a trace particle, that corresponds to its initial data, and the sequence of explicitly defined functionals of a solution of this new Cauchy problem which describe all possible states of a trace particle and an environment.

3 The main result: the generalized Fokker – Planck equation

In view of the fact that every marginal distribution function of initial data (6) is specified by the initial marginal distribution function of a trace particle on allowed configurations, the states given in terms of the sequence $F(t) = (1, F_{1+0}(t, x), F_{1+1}(t, x, x_1), \dots, F_{1+s}(t, x, x_1, \dots, x_s), \dots)$ of marginal distribution functions (8) can be described within the framework of the sequence $F(t | F_1(t)) = (1, F_{1+0}(t, x), F_{1+1}(t, x, x_1 | F_{1+0}(t)), \dots, F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t)), \dots)$ of the marginal functionals of the state $F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t))$, $s \geq 1$, which are explicitly defined with respect to the solution $F_{1+0}(t, x)$ of the evolution equation for a trace particle. We refer to such evolution equation for the marginal distribution function of a trace particle as the generalized Fokker – Planck kinetic equation.

If $t \geq 0$, the marginal distribution function of a trace particle $F_{1+0}(t, x)$ is the solution of the generalized Fokker – Planck kinetic equation

$$\begin{aligned} \frac{\partial}{\partial t} F_{1+0}(t, x) = & -\left\langle \frac{p}{M}, \frac{\partial}{\partial q} \right\rangle F_{1+0}(t, x) + \\ & + \sigma^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_1 d\eta \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \dots dx_{n+1} \langle \eta, \left(\frac{p}{M} - \frac{p_1}{m} \right) \rangle \times \\ & \times (\mathfrak{V}_{1+n}(t, \{\mathfrak{t}^*, 1_-^*\}, 2, \dots, n+1) F_{1+0}(t, q, p^*) - \mathfrak{V}_{1+n}(t, \{\mathfrak{t}, 1_+\}, 2, \dots, n+1) F_{1+0}(t, x)), \end{aligned} \quad (10)$$

$$F_{1+0}(t, x)|_{t=0} = F_{1+0}^0(x). \quad (11)$$

The $(1+n)th$ -order generating evolution operator of the collision integral in kinetic equation (10) is determined by the expansion (the expansion over scattering cumulants of evolution operators (9)):

$$\begin{aligned} \mathfrak{V}_{1+n}(t, \{\mathfrak{t}^\sharp, 1_\mp^\sharp\}, 2, \dots, n+1) F_{1+0}(t, q, p^\sharp) & \doteq \\ & \doteq n! \sum_{k=0}^n (-1)^k \sum_{m_1=1}^n \dots \sum_{m_k=1}^{n-m_1-\dots-m_{k-1}} \frac{1}{(n-m_1-\dots-m_k)!} \times \\ & \times \mathfrak{A}_{1+n-m_1-\dots-m_k}(t, \{\mathfrak{t}^\sharp, 1_\mp^\sharp\}, 2, \dots, 1+n-m_1-\dots-m_k) \times \\ & \times F_{0+1+n-m_1-\dots-m_k}^0(q \mp \sigma\eta, p_1^\sharp, x_2, \dots, x_{1+n-m_1-\dots-m_k}) \prod_{i_1=2}^{1+n-m_1-\dots-m_k} \mathcal{X}_2(q, q_{i_1}) \times \\ & \times \mathfrak{A}_1(t, \mathfrak{t}^\sharp) \prod_{j=1}^k \left(\frac{1}{m_j!} \mathfrak{A}_{1+m_j}(-t, \mathfrak{t}^\sharp, 2+n-m_j-\dots-m_k, \dots, \right. \end{aligned} \quad (12)$$

$$\begin{aligned}
& (1 + n - m_{j+1} - \dots - m_k) F_{0+m_j}^0(x_{2+n-m_j-\dots-m_k}, \dots, x_{1+n-m_{j+1}-\dots-m_k}) \times \\
& \times \prod_{i_2=2+n-m_j-\dots-m_k}^{1+n-m_{j+1}-\dots-m_k} \mathcal{X}_2(q, q_{i_2}) \mathfrak{A}_1(t, \mathfrak{t}^\sharp) F_{1+0}(t, q, p^\sharp),
\end{aligned}$$

where the indices $(\mathfrak{t}^\sharp, 1_\mp^\sharp)$ denote that cumulants (9) of evolution operators (7) act on the phase points (q, p^\sharp) and $(q \mp \sigma\eta, p_1^\sharp)$, respectively.

If $t \leq 0$, the generalized Fokker – Planck kinetic equation takes the form:

$$\begin{aligned}
& \frac{\partial}{\partial t} F_{1+0}(t, x) = -\left\langle \frac{p}{M}, \frac{\partial}{\partial q} \right\rangle F_{1+0}(t, x) + \\
& + \sigma^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_1 d\eta \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \dots dx_{n+1} \langle \eta, \left(\frac{p}{M} - \frac{p_1}{m} \right) \rangle \times \\
& \times (\mathfrak{V}_{1+n}(t, \{\mathfrak{t}, 1_-\}, 2, \dots, n+1) F_{1+0}(t, x) - \mathfrak{V}_{1+n}(t, \{\mathfrak{t}^*, 1_+^*\}, 2, \dots, n+1) F_{1+0}(t, q, p^*)),
\end{aligned} \tag{13}$$

where the generating evolution operators $\mathfrak{V}_{1+n}(t)$, $n \geq 0$, are defined by formula (12).

The marginal functionals of the state $F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t))$ are represented by the following series:

$$\begin{aligned}
& F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t)) \doteq \\
& \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \dots dx_{s+n} \mathfrak{V}_{1+n}(t, \{\mathfrak{t}, Y\}, X \setminus Y) F_{1+0}(t, x),
\end{aligned} \tag{14}$$

where the generating evolution operators $\mathfrak{V}_{1+n}(t, \{\mathfrak{t}, Y\}, X \setminus Y)$, $n \geq 0$, are defined similar to expansions (12) and they will be constructed in next section.

We remark that in terms of marginal functionals of the state (14) the collision integral of the generalized Fokker – Planck kinetic equation (10) is represented in the form:

$$\begin{aligned}
& \mathcal{I}_{GFPE} = \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_1 d\eta \langle \eta, \left(\frac{p}{M} - \frac{p_1}{m} \right) \rangle \times \\
& \times (F_{1+1}(t, q, p^*, q - \sigma\eta, p_1^* | F_{1+0}(t)) - F_{1+1}(t, x, q + \sigma\eta, p_1 | F_{1+0}(t))).
\end{aligned} \tag{15}$$

In case of a one-dimensional system the structure of collision integral (15) was considered in paper [14].

Thus, the objective of this paper is to prove that initial-value problem of the BBGKY hierarchy (1),(6) is equivalent to initial-value problem of the generalized Fokker – Planck equation (10),(11) and a sequence of marginal functionals of the state $F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t))$, $s \geq 1$, defined by series (14).

Finally, we note that the possibility to describe the evolution of all possible states of a many-particle system composed of a trace particle and an environment within the framework of the Cauchy problem of the generalized Fokker – Planck equation and by a sequence of the marginal functionals of the state along with the corresponding Cauchy problem of the BBGKY hierarchy is an inherent property of the description of many-particle systems within the framework of the formalism of nonequilibrium grand canonical ensemble which is adopted to the description of infinite-particle systems in suitable functional spaces [10].

4 Kinetic cluster expansions of cumulants of evolution operators

We introduce the transformation of cumulants (9) which makes possible to represent marginal distribution functions (8) in case of $s \geq 1$ in terms of the expansions with respect to the marginal distribution function of a trace hard sphere, i.e. function (8) in case of $s = 0$.

We expand cumulants (9) of operators (7) into the following kinetic cluster expansions:

$$\begin{aligned} \mathfrak{A}_{1+n}(-t, \{\mathfrak{t}, Y\}, X \setminus Y) F_{0+s+n}^0(x_1, \dots, x_{s+n}) \prod_{i=1}^{s+n} \mathcal{X}_2(q, q_i) F_{1+0}^0(x) = \\ = \sum_{k=0}^n \frac{n!}{(n-k)!k!} \mathfrak{V}_{1+n-k}(t, \{\mathfrak{t}, Y\}, s+1, \dots, s+n-k) \mathfrak{A}_{1+k}(-t, \mathfrak{t}, s+n-k+1, \dots, s+n) F_{0+k}^0(x_{s+n-k+1}, \dots, x_{s+n}) \prod_{i=s+n-k+1}^{s+n} \mathcal{X}_2(q, q_i) F_{1+0}^0(x), \quad n \geq 0, \end{aligned} \quad (16)$$

where $\mathcal{X}_2(q, q_i)$ is the Heaviside step function of the allowed configurations $\mathbb{R}^6 \setminus \mathbb{W}_2$ of two hard spheres and $F(0) = (1, F_{1+0}^0(x), \dots, F_{1+0}^0(x) F_{0+s}^0(x_1, \dots, x_s), \dots)$ is the sequence of initial marginal distribution functions (2). We remark that the structure of cluster expansions (16) is conditioned by an equivalence of methods of the description of states in terms of a solution of the BBGKY hierarchy (1), i.e. by the sequence $F(t) = (1, F_{1+0}(t, x), F_{1+1}(t, x, x_1), \dots, F_{1+s}(t, x, x_1, \dots, x_s), \dots)$, and in terms of the sequence $F(t \mid F_{1+0}(t)) = (1, F_{1+0}(t, x), F_{1+1}(t, x, x_1 \mid F_{1+0}(t)), \dots, F_{1+s}(t, x, x_1, \dots, x_s \mid F_{1+0}(t)), \dots)$, where $F_{1+0}(t)$ is defined by series (8) in case of $s = 0$, and $F_{1+s}(t, x, x_1, \dots, x_s \mid F_{1+0}(t))$, $s \geq 1$, are marginal functionals of the state (14).

We give a few examples of recurrence relations (16)

$$\begin{aligned} \mathfrak{A}_1(-t, \{\mathfrak{t}, Y\}) F_{0+s}^0(x_1, \dots, x_s) \prod_{i=1}^s \mathcal{X}_2(q, q_i) F_{1+0}^0(x) &= \mathfrak{V}_1(t, \{\mathfrak{t}, Y\}) \mathfrak{A}_1(-t, \mathfrak{t}) F_{1+0}^0(x), \\ \mathfrak{A}_2(-t, \{\mathfrak{t}, Y\}, s+1) F_{0+s+1}^0(x_1, \dots, x_{s+1}) \prod_{i=1}^{s+1} \mathcal{X}_2(q, q_i) F_{1+0}^0(x) &= \\ &= \mathfrak{V}_2(t, \{\mathfrak{t}, Y\}, s+1) \mathfrak{A}_1(-t, \mathfrak{t}) F_{1+0}^0(x) + \\ &+ \mathfrak{V}_1(t, \{\mathfrak{t}, Y\}) \mathfrak{A}_2(-t, \mathfrak{t}, s+1) F_{0+1}^0(x_{s+1}) \mathcal{X}_2(q, q_{s+1}) F_{1+0}^0(x). \end{aligned}$$

Solutions of these recurrence relations are given by the following expansions (expansions over scattering operators):

$$\begin{aligned} \mathfrak{V}_1(t, \{\mathfrak{t}, Y\}) &= \mathfrak{A}_1(-t, \{\mathfrak{t}, Y\}) F_{0+s}^0(x_1, \dots, x_s) \prod_{i=1}^s \mathcal{X}_2(q, q_i) \mathfrak{A}_1(t, \mathfrak{t}), \\ \mathfrak{V}_2(t, \{\mathfrak{t}, Y\}, s+1) &= \mathfrak{A}_2(-t, \{\mathfrak{t}, Y\}, s+1) F_{0+s+1}^0(x_1, \dots, x_{s+1}) \prod_{i=1}^{s+1} \mathcal{X}_2(q, q_i) \mathfrak{A}_1(t, \mathfrak{t}) - \\ &- \mathfrak{A}_1(-t, \{\mathfrak{t}, Y\}) F_{0+s}^0(x_1, \dots, x_s) \prod_{i=1}^s \mathcal{X}_2(q, q_i) \mathfrak{A}_1(t, \mathfrak{t}) \mathfrak{A}_2(-t, \mathfrak{t}, s+1) F_{0+1}^0(x_{s+1}) \times \\ &\times \mathcal{X}_2(q, q_{s+1}) \mathfrak{A}_1(t, \mathfrak{t}). \end{aligned}$$

In the general case solutions of recurrence relations (16), i.e. the $(1+n)th$ -order generating evolution operator $\mathfrak{V}_{1+n}(t)$, is given by the expansion ($s \geq 1, n \geq 0$):

$$\begin{aligned}
\mathfrak{V}_{1+n}(t, \{\mathfrak{t}, Y\}, X \setminus Y) &\doteq n! \sum_{k=0}^n (-1)^k \sum_{m_1=1}^n \dots \sum_{m_k=1}^{n-m_1-\dots-m_{k-1}} \frac{1}{(n-m_1-\dots-m_k)!} \times \\
&\times \mathfrak{A}_{1+n-m_1-\dots-m_k}(-t, \{\mathfrak{t}, Y\}, s+1, \dots, s+n-m_1-\dots-m_k) \\
&\times F_{0+s+n-m_1-\dots-m_k}^0(x_1, \dots, x_{s+n-m_1-\dots-m_k}) \prod_{i_1=1}^{s+n-m_1-\dots-m_k} \mathcal{X}_2(q, q_{i_1}) \mathfrak{A}_1(t, \mathfrak{t}) \times \\
&\times \prod_{j=1}^k \left(\frac{1}{m_j!} \mathfrak{A}_{1+m_j}(-t, \mathfrak{t}, s+1+n-m_j-\dots-m_k, \dots, s+n-m_{j+1}-\dots-m_k) \times \right. \\
&\times F_{0+m_j}^0(x_{s+1+n-m_j-\dots-m_k}, \dots, x_{s+n-m_{j+1}-\dots-m_k}) \prod_{i_2=s+1+n-m_j-\dots-m_k}^{s+n-m_{j+1}-\dots-m_k} \mathcal{X}_2(q, q_{i_2}) \mathfrak{A}_1(t, \mathfrak{t}) \Big).
\end{aligned} \tag{17}$$

This statement is proved by induction.

Thus, generating evolution operators (17) of marginal functionals of the state (14) and hence the collision integral of the generalized Fokker – Planck kinetic equation (10) are determined by the initial correlations connected with the forbidden configurations of hard spheres and by the initial state of an environment.

5 The derivation of the Fokker – Planck kinetic equation

Using kinetic cluster expansions (16) of cumulants of operators (9), we derive the generalized Fokker – Planck kinetic equation (10) for a trace hard sphere in an environment which is a system of a non-fixed number of identical hard spheres.

We shall establish that the marginal distribution function defined by series (8),(9) in case of $s = 0$, i.e.

$$\begin{aligned}
F_{1+0}(t, x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \dots dx_n \mathfrak{A}_{1+n}(-t, \mathfrak{t}, 1, \dots, n) F_{1+0}^0(x) \times \\
&\times F_{0+n}^0(x_1, \dots, x_n) \prod_{i=1}^n \mathcal{X}(q, q_i),
\end{aligned} \tag{18}$$

is governed by evolution equation (10) (or (13)).

In view of the validity in the sense of the norm convergence of the space of integrable functions of the following equalities for cumulants of groups (9):

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{1}{t} \mathfrak{A}_1(-t, \mathfrak{t}) f_{1+0}(x) &= \mathcal{L}_{1+0} f_{1+0}(x), \\
\lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \mathfrak{A}_2(-t, \mathfrak{t}, 1) f_{1+1}(x, x_1) &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \mathcal{L}_{\text{int}}(\mathfrak{t}, 1) f_{1+1}(x, x_1), \\
\lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^{3n} \times \mathbb{R}^{3n}} dx_1 \dots dx_n \mathfrak{A}_{1+n}(-t, \mathfrak{t}, 1, \dots, n) f_{1+n} &= 0, \quad n \geq 2,
\end{aligned}$$

where $f_{1+0} \in L_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $f_{1+n} \in L_0^1(\mathbb{R}^{3(1+n)} \times \mathbb{R}^{3(1+n)})$, and the operators \mathcal{L}_{1+0} and $\mathcal{L}_{\text{int}}(\mathbf{t}, 1)$ are defined by formulas (3) and (4), respectively, then as a result of the differentiation over the time variable of expression (18) in the sense of the pointwise convergence on the space $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ we obtain

$$\begin{aligned} \frac{\partial}{\partial t} F_{1+0}(t, x) = -\left\langle \frac{p}{M}, \frac{\partial}{\partial q} \right\rangle F_{1+0}(t, x) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \mathcal{L}_{\text{int}}(\mathbf{t}, 1) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \\ \dots dx_{n+1} \mathfrak{A}_{1+n}(-t, \{\mathbf{t}, 1\}, 2, \dots, n+1) F_{1+0}^0(x) F_{0+n+1}^0(x_1, \dots, x_{n+1}) \prod_{i=1}^{n+1} \mathcal{X}_2(q, q_i). \end{aligned} \quad (19)$$

We represent the second term of the right-hand side of this equality in terms of marginal distribution function (18) of a trace hard sphere. To this end we expand cumulants (9) in series (18) into kinetic cluster expansions (16) for the case $s = 1$. Then we transform the series over the summation index n and the sum over the index k to the two-fold series. As a result the following equality holds:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \dots dx_{n+1} \mathfrak{A}_{1+n}(-t, \{\mathbf{t}, 1\}, 2, \dots, n+1) F_{1+0}^0(x) \times \\ \times F_{0+n+1}^0(x_1, \dots, x_{n+1}) \prod_{i=1}^{n+1} \mathcal{X}_2(q, q_i) = \\ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^{n+k}} dx_2 \dots dx_{n+1+k} \mathfrak{B}_{1+n}(t, \{\mathbf{t}, 1\}, 2, \dots, n+1) \times \\ \times \mathfrak{A}_{1+k}(-t, \mathbf{t}, n+2, \dots, 1+n+k) F_{1+0}^0(x) F_{0+k}^0(x_{2+n}, \dots, x_{1+n+k}) \prod_{i=2+n}^{1+n+k} \mathcal{X}_2(q, q_i). \end{aligned} \quad (20)$$

According to equalities (19) and (20), and taking into account definition (4) of the operator $\mathcal{L}_{\text{int}}(\mathbf{t}, 1)$, from equality (19) for $t \geq 0$, we finally derive

$$\begin{aligned} \frac{\partial}{\partial t} F_{1+0}(t, x) = -\left\langle \frac{p}{M}, \frac{\partial}{\partial q} \right\rangle F_{1+0}(t, x) + \\ + \sigma^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_1 d\eta \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \dots dx_{n+1} \left\langle \eta, \left(\frac{p}{M} - \frac{p_1}{m} \right) \right\rangle \times \\ \times \left(\mathfrak{B}_{1+n}(t, \{\mathbf{t}^*, 1_-^*\}, 2, \dots, n+1) F_{1+0}(t, q, p^*) - \mathfrak{B}_{1+n}(t, \{\mathbf{t}, 1_+\}, 2, \dots, n+1) F_{1+0}(t, x) \right), \end{aligned}$$

where we used notations accepted in equation (10). The collision integral series converges in the sense of the norm convergence of the space $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ under the condition that: $\alpha < e^{-4}$, where $\sup_{n \geq 0} \alpha^{-n} \|F_{0+n}^0\|_{L_n^1} < +\infty$ (the condition on the collision integral coefficients). In next section this fact will be proved in the general case.

We treat the constructed identity for the marginal distribution function of a trace hard sphere as the kinetic equation for a trace hard sphere in an environment of identical hard spheres.

Now we consider the structure of the constructed Fokker – Planck collision integral (15), namely, we consider the first term of its expansion

$$\begin{aligned} \mathcal{I}_{GFPE}^{(0)} = \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_1 d\eta \langle \eta, \left(\frac{p}{M} - \frac{p_1}{m} \right) \rangle & (\mathfrak{V}_1(t, \{\mathfrak{t}^*, 1_-^*\}) F_{1+0}(t, q, p^*) - \\ & - \mathfrak{V}_1(t, \{\mathfrak{t}, 1_+\}) F_{1+0}(t, x)). \end{aligned}$$

Applying the Duhamel equation to the generating evolution operator $\mathfrak{V}_1(t)$

$$\begin{aligned} S_2(-t, \mathfrak{t}, 1) f_2(x, x_1) = S_1(-t, \mathfrak{t}) S_1(-t, 1) f_2(x, x_1) + \\ + \int_0^t d\tau S_1(-t + \tau, \mathfrak{t}) S_1(-t + \tau, 1) \mathcal{L}_{\text{int}}(\mathfrak{t}, 1) S_2(-\tau, \mathfrak{t}, 1) f_2(x, x_1), \end{aligned}$$

where the operator $\mathcal{L}_{\text{int}}(\mathfrak{t}, 1)$ is defined by formula (4) on $f_2 \in L_{1+1,0}^1$, then the expression $\mathcal{I}_{GFPE}^{(0)}$ is represented in the form:

$$\begin{aligned} \mathcal{I}_{GFPE}^{(0)} = \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{S}_{0,+}^2} dp_1 d\eta \langle \eta, \left(\frac{p}{M} - \frac{p_1}{m} \right) \rangle & \left(S_1(-t, 1_-^*) F_{0+1}^0(q - \sigma\eta, p_1^*) F_{1+0}(t, q, p^*) - \right. \\ & - S_1(-t, 1_+) F_{0+1}^0(q + \sigma\eta, p_1) F_{1+0}(t, x) + \int_0^t d\tau (S_1(-t + \tau, \mathfrak{t}^*) S_1(-t + \tau, 1_-^*) \times \\ & \times \mathcal{L}_{\text{int}}(\mathfrak{t}^*, 1_-^*) S_2(-\tau, \mathfrak{t}^*, 1_-^*) F_{0+1}^0(q - \sigma\eta, p_1^*) S_1(t, \mathfrak{t}^*) F_{1+0}(t, q, p^*) - \\ & \left. - S_1(-t + \tau, \mathfrak{t}) S_1(-t + \tau, 1_+) \mathcal{L}_{\text{int}}(\mathfrak{t}, 1_+) S_2(-\tau, \mathfrak{t}, 1_+) F_{0+1}^0(q + \sigma\eta, p_1) S_1(t, \mathfrak{t}) F_{1+0}(t, x)) \right). \end{aligned}$$

Thus, the first term of the collision integral $\mathcal{I}_{GFPE}^{(0)}$ of the generalized Fokker – Planck equation coincides with the collision integral of the Fokker – Planck equation established by N.N. Bogolyubov [1] within the framework of the perturbation theory.

We remark that in the space homogeneous case the Markovian approximation of the Fokker – Planck collision integral has a more general structure then the canonical collision integral of the Fokker – Planck equation [4].

Let $F_{1+0}^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and the initial distribution functions of an environment such that $\sup_{n \geq 0} \alpha^{-n} \|F_{0+n}^0\|_{L_n^1} < +\infty$, where $\alpha > 0$ is a parameter (it is interpreted as density). Then for a solution of the Cauchy problem of the generalized Fokker – Planck equation (10),(11) in the space of integrable functions $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ the following statement is true.

Theorem 1. *If $\alpha < e^{-4}$, for $t \in \mathbb{R}$ a solution of the Cauchy problem of the generalized Fokker – Planck equation (10),(11) ((13),(11)) is determined by the series:*

$$\begin{aligned} F_{1+0}(t, x) = \\ = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \dots dx_n \mathfrak{A}_{1+n}(-t) F_{1+0}^0(x) F_{0+n}^0(x_1, \dots, x_n) \prod_{i=1}^n \mathcal{X}_2(q, q_i), \end{aligned} \quad (21)$$

where the generating operators $\mathfrak{A}_{1+n}(-t)$, $n \geq 0$, are cumulants of groups (7) defined by (9). For initial data $F_{1+0}^0 \in L_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $F_{0+n}^0 \in L_0^1(\mathbb{R}^{3n} \times \mathbb{R}^{3n})$ it is a strong (classical) solution and for an arbitrary initial data $F_{1+0}^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $F_{0+n}^0 \in L^1(\mathbb{R}^{3n} \times \mathbb{R}^{3n})$ it is a weak (generalized) solution.

The scheme of the proof of this theorem is similar to the proof of an existence theorem for the generalized Enskog kinetic equation [15].

6 Marginal functionals of the state

Using kinetic cluster expansions (16), we represent solution expansions (8) of the BBGKY hierarchy (1) in case of $s \geq 1$, in the form of the expansions with respect to marginal distribution function (18) which is governed by the derived Fokker – Planck equation (10).

In case of $s \geq 1$ in every term of series (8) we expand cumulants of groups (9) into kinetic cluster expansions (16). As a result of the transformation of the series over the summation index n and the sum over the index k to the two-fold series we obtain the following equality:

$$\begin{aligned}
F_{1+s}(t, x, x_1, \dots, x_s) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \dots dx_{s+n} \mathfrak{A}_{1+n}(-t, \{\mathfrak{t}, Y\}, X \setminus Y) F_{1+0}^0(x) \times \\
&\times F_{0+s+n}^0(x_1, \dots, x_{s+n}) \prod_{i=1}^{s+n} \mathcal{X}_2(q, q_i) = \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \dots dx_{s+n} \mathfrak{B}_{1+n}(t, \{\mathfrak{t}, Y\}, X \setminus Y) \sum_{k=0}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^k} dx_{s+n+1} \dots \\
&\dots dx_{s+n+k} \mathfrak{A}_{1+n}(-t, \mathfrak{t}, s+n+1, \dots, s+n+k) F_{1+0}^0(x) \times \\
&\times F_{0+k}^0(x_{s+n+1}, \dots, x_{s+n+k}) \prod_{i=s+n+1}^{s+n+k} \mathcal{X}_2(q, q_i).
\end{aligned}$$

Taking into account the definition of marginal distribution function (18) of a trace hard sphere in the obtained expression, finally we establish the equality:

$$\begin{aligned}
F_{1+s}(t, x, x_1, \dots, x_s) &= \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \dots dx_{s+n} \mathfrak{B}_{1+n}(t, \{\mathfrak{t}, Y\}, X \setminus Y) F_{1+0}(t, x) = \\
&= F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t)), \quad s \geq 1,
\end{aligned}$$

where the generating evolution operators $\mathfrak{B}_{1+n}(t)$, $n \geq 0$, are defined by formula (17) as solutions of recurrence relations (16).

We establish the existence of marginal functionals of the state (14) for $F_{1+0}^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and initial data of an environment such that: $c \equiv \sup_{n \geq 0} \alpha^{-n} \|F_{0+n}^0\|_{L_n^1} < +\infty$, where $\alpha > 0$ is a parameter which is interpreted as density of an environment.

Owing to the fact that for cumulants of groups (9) the estimate holds

$$\begin{aligned}
&\int_{(\mathbb{R}^3 \times \mathbb{R}^3)^{1+s+n}} dx dx_1 \dots dx_{s+n} |\mathfrak{A}_{1+n}(-t, \{\mathfrak{t}, Y\}, X \setminus Y) f_{1+s+n}(x, x_1, \dots, x_{s+n})| \leq \\
&\leq n! e^{n+2} \|f_{1+s+n}\|_{L_{1+s+n}^1},
\end{aligned}$$

then for the $(1+n)th$ -order generating evolution operator (17) the following inequality is true:

$$\int_{(\mathbb{R}^3 \times \mathbb{R}^3)^{1+s+n}} dx dx_1 \dots dx_{s+n} |\mathfrak{B}_{1+n}(t, \{\mathfrak{t}, Y\}, X \setminus Y) F_{1+0}(t, x)| \leq$$

$$\begin{aligned}
&\leq n!c^2\alpha^s \|F_{1+0}(t)\|_{L_1^1} \sum_{k=0}^n \sum_{m_1=1}^n \dots \sum_{m_k=1}^{n-m_1-\dots-m_{k-1}} e^{n-m_1-\dots-m_k+2} \times \\
&\times \alpha^{n-m_1-\dots-m_k} \prod_{j=1}^k e^{m_j+2} \alpha^{m_j} = \\
&= n!c^2\alpha^s \|F_{1+0}(t)\|_{L_1^1} e^{n+2} \alpha^n \sum_{k=0}^n e^{2k} \sum_{m_1=1}^n \dots \sum_{m_k=1}^{n-m_1-\dots-m_{k-1}} 1.
\end{aligned}$$

As a result of the validity of this inequality and the following estimate:

$$\sum_{m_1=1}^n \dots \sum_{m_k=1}^{n-m_1-\dots-m_{k-1}} 1 = \frac{(n-k+1)\dots(n-1)n}{k!} \leq \frac{n^k}{k!} \leq e^n,$$

for marginal functionals of the state (14) the estimate holds:

$$\begin{aligned}
\|F_{1+s}(t | F_{1+0}(t))\|_{L_{1+s}^1} &\leq \|F_{1+0}(t)\|_{L_1^1} c^2 e^2 \alpha^s \sum_{n=0}^{\infty} e^{2n} \alpha^n \sum_{k=0}^n e^{2k} = \\
&= \|F_{1+0}(t)\|_{L_1^1} c^2 e^2 \alpha^s \sum_{n=0}^{\infty} e^{2n} \alpha^n \frac{1-e^{2(n+1)}}{1-e^2} \leq \|F_{1+0}(t)\|_{L_1^1} c^2 e^3 \alpha^s \sum_{n=0}^{\infty} (e^4 \alpha)^n.
\end{aligned}$$

Hence, functionals (14) exist and are represented by converged series provided that: $\alpha < e^{-4}$.

Thus, in fact we have proved above that marginal distribution functions (8) in case of $s \geq 1$ and the marginal functionals of the state (14) are equivalent if and only if the generating evolution operators $\mathfrak{V}_{1+n}(t, \{Y\}, X \setminus Y)$, $n \geq 0$, satisfy recurrence relations (16).

We note that the average values of observables are determined by marginal functionals of the state (14). For example, the average value of the $(1+s)$ -ary marginal observable $B^{(1+s)} = (0, \dots, 0, b_{1+s}(x, x_1, \dots, x_{1+s}), 0, \dots)$ is defined by the formula

$$\begin{aligned}
\langle B^{(1+s)} \rangle(t) &= \\
&= \frac{1}{(1+s)!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^{1+s}} dx dx_1 \dots dx_s b_{1+s}(x, x_1, \dots, x_s) F_{1+s}(t, x, x_1, \dots, x_s | F_{1+0}(t)),
\end{aligned}$$

where the function $F_{1+0}(t, x)$ is a solution of the Cauchy problem of the generalized Fokker – Planck equation (10),(11) ((13),(11)).

We emphasize that in fact constructed functionals of a solution of the generalized Fokker – Planck kinetic equation (14) characterize all possible correlations which are created in the process of the evolution a trace hard sphere in an environment.

Thus, in the last two sections we proved the main result of the work, namely, if initial data is specified by distribution functions (6), then all possible states of a trace hard sphere in an environment at arbitrary moment of time can be described within the framework of marginal distribution function of a trace hard sphere governed by the generalized Fokker – Planck equation (10) and the explicitly defined functionals of this function (14) without any approximations.

7 Conclusion

For a many-particle system composed of a trace hard sphere and an environment which is a system of a non-fixed number of identical hard spheres we prove an equivalence of the description of the evolution of states by the Cauchy problem of the BBGKY hierarchy (1),(6) and by the Cauchy problem of the generalized Fokker – Planck kinetic equation (10),(11) and constructed marginal functionals of its solution (14). Thus, the stated Fokker – Planck kinetic equation (10) is the basis of an alternative approach to the description of the evolution of a trace particle in an environment.

We remark that in order to describe the evolution of a trace particle in infinite-particle environment we must to construct a solution of the generalized Fokker – Planck equation (10) for initial data of an environment that belongs to the more general Banach spaces than the space of integrable functions. In that case every term of solution expansion (21) as well as marginal functionals of the state (14) contains the divergent integrals. The stated structure of generating evolution operators of mentioned series makes it possible to regularize the corresponding divergent expressions [14].

The developed approach is related to the problem of a rigorous derivation of the non-Markovian kinetic equation from underlying many-particle dynamics which makes possible to describe the memory effects of the diffusion processes. The specific Fokker – Planck-type kinetic equations can be derived from the constructed generalized Fokker – Planck kinetic equation in the appropriate scaling limits or as a result of certain approximations.

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